

# Stochastic stability and stabilization of a class of state-dependent jump linear systems\*

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## Abstract

This paper deals a continuous-time state-dependent jump linear system, a particular kind of stochastic switching system. In particular, we consider a situation when the transition rate of the random jump process depends on the state variable, and addressed the problem of stochastic stability and stabilization analysis for the proposed system. Numerically solvable sufficient conditions for the stochastic stability and stabilization of the proposed system is established in terms of linear matrix inequalities. The obtained results are illustrated in numerical examples.

## 1 Introduction

Systems subject to random abrupt changes can be modeled by Random Jump linear systems (RJLS) such as manufacturing systems, networked control systems, economics and finance *etc.* RJLS are a special class of hybrid systems, typically, described by a set of classical differential (or difference) equations and a random jump process governing the jumps among them.

When the random jump process of RJLS is assumed to be a finite state time-homogeneous Markovian process with a known transition rate (or probability), then this particular class of systems are widely known as Markov jump linear systems (MJLS) in the literature. The theory of stability, optimal and robust control, as well as important applications of MJLS can be found for instance in [6], [14], [11], [9], [5] and the references therein. In general, the studies of MJLS assume that the underlying random process is time-homogeneous Markov, which is quite a restrictive assumption.

In this article, we consider the analysis of RJLS where the random jump process depends on the state variable. Such class of systems are referred to as “state-dependent jump linear systems (SDJLS)” in this article. The given problem is motivated by following scenarios. In fault tolerant control systems, the failure rate of a component generally depends on its age, wear, accumulated stress *etc.* It is reasonable to assume that the failure rate of a component at time  $t$  depends on the state of the component at age  $t$ , see for example [1]. In this case, state variable may be a measure of wear, accumulated stress of component *etc.*, which affect its failure rate. As an another scenario, consider a case of stock market with two regimes: up and down. The transition rate between the regimes usually result from the state of economy, the general mood of the investors in the market *etc.*, which can be regarded in general as the state of the market. Also, for instance, in [8], the authors dealt with the problem of describing the underlying reasons for the failure mechanism of a process and modelled the degradation or wear of the process as a Markov process whose transitions depend on the state of the process. As an application, the wear of cylinder lines in a heavy-duty marine diesel engines is considered as a state-dependent Markov process. Also, let us consider a modelling of macroeconomic and financial time series. In [10], a regime-switching model for the sample path of a time series is examined, where the transition probabilities between the regimes depend on the state variable. One can find more examples or scenarios of this kind in the literature.

To the best of the authors’ knowledge, only a few works have been carried out on stability and control of SDJLS. In [18], a study of hybrid switching diffusion processes, a kind of state-dependent jump non-linear systems, has been carried out by treating existence, uniqueness, stability of the solutions *etc.* In [16], the authors considered that the transition rate of the random jump process depends on both the state variable and the control input in such a way that both the state variable and the control input affect the time scale of the random jump process, thus affecting its transition

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rate, and obtained a control policy for a given functional using stochastic maximum principle. A model for planning and maintenance in flexible manufacturing system is proposed in [3], where the failure rate of a machine depends on the state variable, and computed an optimal control using dynamic programming. In [7], a two-time scale model of production plant is considered as a jump diffusion model where the failure rate of a machine depends on the state variable, and obtained an optimal control.

In this article, we consider the state-dependent transition rates explicitly as: the transition rates vary depending on which set the state of the system belongs to. This is a reasonable assumption because the state of the system at any time belong to one of the predefined sets, and the transition rate can be considered to have different values across the predefined sets. The major difference of the current work in this article with the existing literature on RJLS is that the random jump process does not follow Markov property. Under the given assumption that the transition rates vary depending on the set to which the state of the system belongs to, we prove that the times at which the change of transition rates occur are stopping times and accordingly we consider a Dynkin's formula with stopping. Utilizing this formalism, we obtained numerically tractable sufficient conditions for stochastic stability and stabilization in terms of linear matrix inequalities (LMIs), though for a restricted class of SDJLS described in section 2.

The rest of the article is organized as follows: section 2 gives the description of a mathematical model of the SDJLS studied in this article. In section 3, sufficient conditions for the stochastic stability and stabilization of the SDJLS are obtained. In section 4, numerical examples are given to illustrate the proposed results, and the concluding remarks are addressed in section 5.

*Notation* : Let  $\mathbb{R}^n$  be the  $n$ -dimensional real Euclidean space.  $A^T$  is the transpose of a matrix  $A$ .  $\lambda_{\min}(A)$  represent the minimum eigenvalue of a matrix  $A$ . Given two matrices  $L$  and  $M$ ,  $L \succ M$  (or  $L \prec M$ ) denotes that the matrix  $L - M$  is positive definite (or negative definite). The standard vector norm in  $\mathbb{R}^n$  is indicated by  $\|\cdot\|$  the corresponding induced norm of a matrix  $A$  by  $\|A\|$ . Let  $\mathbb{Q}$  be the set of rational numbers and  $\mathbb{N}_0$  be the set of natural numbers including 0. The operator  $\cup$  denotes the union and  $\cap$  denotes intersection. Given two sets  $A, B$ ,  $A \setminus B$  denotes the set  $A \cap B^c$ . The empty set is represented by  $\phi$ . For any  $a, b \in \mathbb{R}$ ,  $a \wedge b$  represents the minimum of two numbers  $a, b$ .  $I_A(x)$  is the standard indicator function which has a value 1 if  $x \in A$ , otherwise has a value 0. The mathematical expectation of a random variable  $X$  is denoted by  $\mathbb{E}[X]$ . Let  $g(X_t)$  be an arbitrary functional of a stochastic process  $X_t$ ; denote  $\mathbb{E}[g(X_t)]_{X_t=X}$  as the expectation of the functional  $g(X_t)$  at  $X_t = X$ .

## 2 Mathematical Model

Consider a SDJLS in a fixed probability space  $(\Omega, \mathcal{F}, Pr)$

$$\begin{aligned}\dot{x}(t) &= A_{\theta(t)}x(t), \\ x(0) &= x_0,\end{aligned}\tag{1}$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $x_0 \in \mathbb{R}^n$  is the initial state,  $A_{\theta(t)} \in \mathbb{R}^{n \times n}$  be system matrices which depend on  $\theta(t)$ . Let  $\theta(t) \in S := \{1, 2, \dots, N\}$ , describing the mode of the system at time  $t$ , be a finite space continuous time jump process whose transitions depends on the state variable  $x(t)$  as follows, for  $i \neq j$  :

$$Pr\{\theta(t+h) = j / \theta(t) = i, x(t)\} = \begin{cases} \lambda_{ij}^1 h + o(h), & \text{if } x(t) \in C_1, \\ \lambda_{ij}^2 h + o(h), & \text{if } x(t) \in C_2, \\ \vdots \\ \lambda_{ij}^K h + o(h), & \text{if } x(t) \in C_K, \end{cases}\tag{2}$$

where  $h > 0$  and  $\lim_{h \rightarrow 0} \frac{o(h)}{h} = 0$ . Let  $\mathcal{K} \triangleq \{1, 2, \dots, K\}$ . We assume that  $C_1, C_2, \dots, C_K \subseteq \mathbb{R}^n$ ,  $C_1 \cup C_2 \cup \dots \cup C_K = \mathbb{R}^n$  and  $C_i \cap C_j = \phi$  for any  $i \neq j \in \mathcal{K}$ . For each  $m \in \mathcal{K}$ ,  $\lambda_{ij}^m$  is the transition rate of  $\theta(t)$  from mode  $i$  to mode  $j$  with  $\lambda_{ij}^m \geq 0$  for  $i \neq j$ , and  $\lambda_{ii}^m = -\sum_{j=1, j \neq i}^N \lambda_{ij}^m$ . It represents the probability per time unit that  $\theta(t)$  makes a transition from mode  $i$  to mode  $j$ .  $o(h)$  is little- $o$  notation defined by  $\lim_{h \rightarrow 0} \frac{o(h)}{h} = 0$ . A trivial remark here is that  $m \in \mathcal{K}$  in  $\lambda_{ij}^m$  as per (2) is a notation followed in this article, but one should not get confused with the actual power of transition rates.

From the assumption  $C_1 \cup C_2 \cup \dots \cup C_K = \mathbb{R}^n$  and  $C_i \cap C_j = \emptyset$  for any  $i \neq j \in \mathcal{K}$ , at any time  $t$ ,  $x(t)$  belongs to one of the sets  $C_l$ ,  $l \in \mathcal{K}$ , accordingly the transition rate of  $\theta(t)$  is  $\lambda_{ij}^l$  from (2). Observe that the transition rate of  $\theta(t)$  depends on the state variable  $x(t)$ , hence we call the system (1) as SDJLS.

We slightly change the notations of the SDJLS (1) and the mode  $\theta(t)$  (2) such that the dealing of the state dependence becomes simpler. For this purpose consider  $\sigma_t \in \mathcal{K}$ , which provide the information of state variable  $x(t)$  at each time  $t$  as

$$\sigma_t = \begin{cases} 1, & \text{if } x(t) \in C_1, \\ 2, & \text{if } x(t) \in C_2, \\ \vdots & \\ K, & \text{if } x(t) \in C_K. \end{cases} \quad (3)$$

Let  $r(\sigma_t, t) \in S$  (which is equivalent to  $\theta(t)$ ), denote the mode of the system at time  $t$ , be a finite space continuous time jump process whose transitions depends on  $\sigma_t$ . Implicitly,  $r(\sigma_t, t)$  depends on the state variable  $x(t)$  as follows, for  $i \neq j$ ,

$$Pr\{r(\sigma_{t+h}, t+h) = j / r(\sigma_t, t) = i\} = \begin{cases} \lambda_{ij}^1 h + o(h), & \text{if } \sigma_t = 1, \\ \lambda_{ij}^2 h + o(h), & \text{if } \sigma_t = 2, \\ \vdots & \\ \lambda_{ij}^K h + o(h), & \text{if } \sigma_t = K, \end{cases} \quad (4)$$

where  $\lambda_{ij}^l$ , for  $l \in \mathcal{K}$  is defined in (2)

Accordingly, we can describe the SDJLS (1) as

$$\begin{aligned} \dot{x}(t) &= A_{r(\sigma_t, t)} x(t), \\ x(0) &= x_0, \end{aligned} \quad (5)$$

where  $A_{r(\sigma_t, t)} \in \mathbb{R}^{n \times n}$  be system matrices (which are equivalent to  $A_{\theta(t)}$ ) which depend on  $r(\sigma_t, t)$ . From now onwards, we analyse the system (5) with jump process (4), which is equivalent to analysing the system (1) with jump process (2).

*Remark 1.* One can observe that the overall system (5) is nonlinear due to the presence of jump process  $r(\sigma_t, t)$ . The existence and uniqueness of solution to the system (5) follows directly from theorem 2.1 of [18].

*Remark 2.* Observe that, conditioning on  $r(\sigma_t, t) = i$ ,  $r(\sigma_{t+h}, t+h)$  depends on  $x(t)$  for any  $h > 0$ , and from (5), which in turn depends on  $r(\sigma_s, s)$ ,  $s < t$ . Hence  $r(\sigma_t, t)$  is not a Markov process. However  $(x(t), r(\sigma_t, t), \sigma_t)$  is a joint Markov process. This point is stated and proved in the following lemma.

**Lemma 1.**  $(x(t), r(\sigma_t, t), \sigma_t)$  is a joint Markov process.

*Proof.* Given in the Appendix. □

The solution to (5) can be constructed as presented in the sequel. In that direction, we define first exit times from the sets  $C_j$ , for  $j \in \mathcal{K}$ . We use a convention  $\inf \emptyset \triangleq \infty$ .

- Step 0: Let  $x(0) \in C_{i_0}$ , where  $i_0 \in \mathcal{K}$ . Define  $\tau_0$  as the first exit time from  $C_{i_0}$  as

$$\tau_0 = \inf\{t \geq 0 : \Phi_{i_0}(t, 0)x(0) \notin C_{i_0}\}.$$

- Step 1: Let  $x(\tau_0) \in C_{i_1}$ , where  $i_1 \neq i_0$ ,  $i_1 \in \mathcal{K}$ . Define  $\tau_1$  as the first exit time from  $C_{i_1}$  after  $\tau_0$  as

$$\tau_1 = \inf\{t \geq \tau_0 : \Phi_{i_1}(t, \tau_0)\Phi_{i_0}(\tau_0, 0)x(0) \notin C_{i_1}\}.$$

- Step 2: Let  $x(\tau_1) \in C_{i_2}$ , where  $i_2 \neq i_1$ ,  $i_2 \in \mathcal{K}$ . Define  $\tau_2$  as the first exit time from  $C_{i_2}$  after  $\tau_1$  as

$$\tau_2 = \inf\{t \geq \tau_1 : \Phi_{i_2}(t, \tau_1)\Phi_{i_1}(\tau_1, \tau_0)\Phi_{i_0}(\tau_0, 0)x(0) \notin C_{i_2}\}.$$

In general, at any step  $m$ , given  $\tau_{m-1}, i_{m-1} \in \mathcal{K}$  of the previous step  $m-1$  which is defined in a similar manner above,

- Step  $m$ : Let  $x(\tau_{m-1}) \in C_{i_m}$ , where  $i_m \neq i_{m-1}, i_m \in \mathcal{K}$ . Define  $\tau_m$  as the first exit time from  $C_{i_m}$  after  $\tau_{m-1}$  as

$$\tau_m = \inf\{t \geq \tau_{m-1} : \Phi_{i_m}(t, \tau_{m-1})\Phi_{i_{m-1}}(\tau_{m-1}, \tau_{m-2}) \cdots \Phi_{i_0}(\tau_0, 0)x(0) \notin C_{i_m}\}, \quad (6)$$

where the random flows  $\Phi(\cdot, \cdot)$  are defined in the sequel. We describe, in general, one of the random flows  $\Phi_{i_m}(t, \tau_{m-1})$  in (6), at step  $m$ , with  $x(\tau_{m-1}) \in C_{i_m}$ , using which any random flow of (5) can be described in a similar fashion. At step  $m$ , during the interval  $[\tau_{m-1}, \tau_m)$ , with  $\tau_{-1} \triangleq 0$ ; let  $n_m \in \mathbb{N}_0$  be the number of regime transitions of  $r(\sigma_t, t)$ ; let  $\{r_0^m, r_1^m, \dots, r_{n_m}^m\} \in S$  be the sequence of regimes visited by  $r(\sigma_t, t)$ ; let  $\{T_0^m, T_1^m, \dots, T_{n_m}^m\} \in [\tau_{m-1}, \tau_m)$  be the successive sojourn times of  $r(\sigma_t, t)$ , which are independent exponentially distributed random variables with parameter  $\lambda_{ij}^{i_m}$ . Let  $S_{n_m}^m \triangleq \sum_{l=0}^{n_m-1} T_l^m$ . Then  $\Phi_{i_m}(t, \tau_{m-1})$  is given by,

$$\Phi_{i_m}(t, \tau_{m-1}) = \begin{cases} e^{A_{r_{n_m}^m}(t - S_{n_m}^m - \tau_{m-1})} e^{A_{r_{n_m-1}^m} T_{n_m-1}^m} \cdots e^{A_{r_1^m} T_1^m} e^{A_{r_0^m} T_0^m}, & \text{if } n_m \geq 1, \\ e^{A_{r_{n_m}^m}(t - \tau_{m-1})}, & \text{if } n_m = 0. \end{cases}$$

*Remark 3.* Notice that, from step 0, step 1,  $\dots$  step  $m, \dots$ ,  $\sigma_t$  and  $r(\sigma_t, t)$  can be described alternatively for  $t \geq 0$  by

$$\sigma_t = \begin{cases} i_0, & \text{if } t \in [0, \tau_0), \\ i_1, & \text{if } t \in [\tau_0, \tau_1), \\ \vdots & \\ i_m, & \text{if } t \in [\tau_{m-1}, \tau_m), \\ \vdots & \end{cases}$$

and

$$Pr\{r(\sigma_{t+h}, t+h) = j / r(\sigma_t, t) = i\} = \begin{cases} \lambda_{ij}^{i_0} h + o(h), & \text{if } t \in [0, \tau_0), \\ \lambda_{ij}^{i_1} h + o(h), & \text{if } t \in [\tau_0, \tau_1), \\ \vdots & \\ \lambda_{ij}^{i_m} h + o(h), & \text{if } t \in [\tau_{m-1}, \tau_m), \\ \vdots & \end{cases}$$

where  $\{i_0, i_1, \dots, i_m, \dots\} \in \mathcal{K}$  and  $i \neq j, i, j \in S$ .

*Remark 4.* Though the alternative reformulations of  $\sigma_t$  and  $r(\sigma_t, t)$  in remark 3 seems not much useful at this point, but the results of section 3 will be based on these reformulations.

*Remark 5.* From step 0, given  $x(0) \in C_{i_0}$ , for  $i_0 \in \mathcal{K}$ , if  $\tau_0 = \infty$ , then  $x(t) \in C_{i_0}$  for all  $t \geq 0$ . In this case the overall system (5) is equivalent to time-homogeneous MJLS with jump process being time-homogeneous Markov with parameter  $\lambda_{ij}^{i_0}$ .

We define the stopping time in the sequel and prove that the first exit times  $\tau_0, \tau_1, \tau_2, \dots$  are the stopping times.

**Definition 1.** Let  $(\Omega, \mathcal{F}, \mathcal{G}_t, Pr)$  be a filtered probability space, then a random variable  $\tau : \Omega \rightarrow [0, \infty]$  (it may take the value  $\infty$ ) is called a stopping time if  $\{\tau \leq t\} \in \mathcal{G}_t$  for any  $t \geq 0$ , i.e; the event  $\{\tau \leq t\}$  is  $\mathcal{G}_t$ -measurable, which implies the event  $\{\tau \leq t\}$  is completely determined by the knowledge of  $\mathcal{G}_t$ .

The following lemma shows that the first exit times given above are in fact stopping times.

**Lemma 2.** The first exit times  $\tau_0, \tau_1, \tau_2, \dots$  described in step 0, step 1, step 2,  $\dots$  are stopping times.

*Proof.* Given in the Appendix. □

Based on lemma 1 and lemma 2, we provide a Dynkin's formula that will be used in the next section.

**Definition 2.** Let  $(x(t), r(\sigma_t, t), \sigma_t)$  be a Markov process and  $\tau_0, \tau_1, \tau_2, \dots$  are stopping times. Let  $\xi(t) \triangleq (x(t), r(\sigma_t, t), \sigma_t)$ . For any suitable Lyapunov function  $V(\xi(t))$ , the Dynkin's formula can be described as [15], [17],

$$\begin{aligned} \mathbb{E}[V(\xi(t))|\xi(0)] - V(\xi(0)) &= \mathbb{E} \left[ \sum_{j=0}^{j^*} \int_{t \wedge \tau_{j-1}}^{t \wedge \tau_j} \mathcal{L}V(\xi(s)) ds | \xi(s) \right] \\ &= \sum_{j=0}^{j^*} \mathbb{E} \left[ \int_{t \wedge \tau_{j-1}}^{t \wedge \tau_j} \mathcal{L}V(\xi(s)) ds | \xi(s) \right], \end{aligned} \quad (7)$$

where  $\mathcal{L}V(\xi(t))$  is the infinitesimal generator of  $V(\xi(t))$ . Here  $\tau_{-1} = 0$ , and  $j = 0, 1, \dots, j^*$ , where  $j^* \in [0, \infty]$  and  $\tau_{j^*} \leq \infty$ .

In general,  $\mathcal{L}V(\xi(t))$  can be understood as the average time rate of change of the function  $V(\xi(t))$  given  $\xi(t)$  at time  $t$ . Also observe that, since  $\xi(t)$  is Markov process, for any  $t \geq 0$ , the expectation terms in (7) are conditioned on  $\xi(t)$ , instead of the natural filtration of  $\xi(t)$  on the interval  $[0, t]$ .

### 3 Main Results

In this section, we present sufficient conditions for stochastic stability and stabilization of the system (5).

#### 3.1 Stochastic stability

We begin with a definition of stochastic stability,

**Definition 3.** For system (5), the equilibrium point 0 is stochastically stable if, for any  $x_0 \in \mathbb{R}^n$  and any  $r(\sigma_0, 0) \in S := \{1, 2, \dots, N\}$  and  $\sigma_0 \in \mathcal{K} := \{1, 2, \dots, K\}$ ,

$$\mathbb{E} \left[ \int_0^\infty \|x(t)\|^2 dt \right] < \infty.$$

We now provide a sufficient condition for stochastic stability.

**Theorem 1.** The system (5) is stochastically stable if there exist positive definite matrices  $P_i \succ 0$ ,  $W_{\kappa i} \succ 0$  for all  $i \in S$  and for all  $\kappa \in \mathcal{K}$ , satisfying

$$A_i^T P_i + P_i A_i + \sum_{j=1}^N \lambda_{ij}^\kappa P_j = -W_{\kappa i}, \quad (8)$$

where  $\lambda_{ij}^\kappa$  is defined in (2).

*Proof:* Consider a  $V(x(t), r(\sigma_t, t), \sigma_t) = x^T(t) P_{r(\sigma_t, t)} x(t)$ , which is quadratic and positive in  $x(t)$ , hence a Lyapunov candidate function. Let the infinitesimal generator of  $V(x(t), r(\sigma_t, t), \sigma_t)$ , for any  $i \in S$  and for any  $\kappa \in \mathcal{K}$ , be given by,

$$\begin{aligned} \mathcal{L}V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) &= \lim_{h \rightarrow 0} \frac{1}{h} \left\{ \mathbb{E} \left[ V(x(t+h), r(\sigma_{t+h}, t+h), \sigma_{t+h}) | (x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) \right] \right. \\ &\quad \left. - V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) \right\} \\ &= x^T(t) \left[ A_i^T P_i + P_i A_i + \sum_{j=1}^N \lambda_{ij}^\kappa P_j \right] x(t). \end{aligned}$$

The above derivation is quite straight forward and follows the similar approach as given in [2], [4] for example. Hence, by (8),

$$\begin{aligned}\mathcal{L}V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) &= -x^T(t)W_{\kappa i}x(t) \\ &\leq -\min_{\kappa \in \mathcal{K}, i \in S} \{\lambda_{\min}(W_{\kappa i})\} x^T(t)x(t).\end{aligned}\quad (9)$$

From (7), consider for any  $i_0 \in \mathcal{K}$ ,

$$\begin{aligned}&\mathbb{E}\left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0)\right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\ &= \sum_{j=0}^{j^*} \mathbb{E}\left[\int_{t \wedge \tau_{j-1}}^{t \wedge \tau_j} \mathcal{L}V(x(s), r(\sigma_s, s), \sigma_s) ds | (x(s), r(\sigma_s, s), \sigma_s)\right],\end{aligned}$$

where  $j^*$ ,  $\tau_{-1}$  and  $\tau_{j^*}$  are given in (7). Let  $\{i_0, i_1, i_2, \dots\} \in \mathcal{K}$  be the successive states visited by  $\sigma_t$  similar to remark 3. Then

$$\begin{aligned}&\mathbb{E}\left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0)\right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\ &= \mathbb{E}\left[\int_0^{\tau_0} \mathcal{L}V(x(s), r(\sigma_s = i_0, s), \sigma_s = i_0) ds | (x(s), r(\sigma_s = i_0, s), \sigma_s = i_0)\right] \\ &\quad + \mathbb{E}\left[\int_{\tau_0}^{\tau_1} \mathcal{L}V(x(s), r(\sigma_s = i_1, s), \sigma_s = i_1) ds | (x(s), r(\sigma_s = i_1, s), \sigma_s = i_1)\right] \\ &\quad + \dots + \mathbb{E}\left[\int_{t \wedge \tau_{j^*-1}}^{t \wedge \tau_{j^*}} \mathcal{L}V(x(s), r(\sigma_s, s), \sigma_s) ds | (x(s), r(\sigma_s, s), \sigma_s)\right].\end{aligned}$$

By (9),

$$\begin{aligned}&\mathbb{E}\left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0)\right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\ &\leq -\min_{\kappa \in \mathcal{K}, i \in S} \{\lambda_{\min}(W_{\kappa i})\} \left( \mathbb{E}\left[\int_0^{\tau_0} x^T(s)x(s) ds\right] + \mathbb{E}\left[\int_{\tau_0}^{\tau_1} x^T(s)x(s) ds\right] + \dots \right. \\ &\quad \left. + \mathbb{E}\left[\int_{t \wedge \tau_{j^*-1}}^{t \wedge \tau_{j^*}} x^T(s)x(s) ds\right] \right).\end{aligned}$$

By denoting  $\sum_{j=0}^{j^*} \int_{t \wedge \tau_{j-1}}^{t \wedge \tau_j} = \int_0^t$ , one obtains,

$$\begin{aligned}&\mathbb{E}\left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0)\right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\ &\leq -\min_{\kappa \in \mathcal{K}, i \in S} \{\lambda_{\min}(W_{\kappa i})\} \mathbb{E}\left[\int_0^t x^T(s)x(s) ds\right].\end{aligned}$$

By rearranging the terms,

$$\begin{aligned}&\min_{\kappa \in \mathcal{K}, i \in S} \{\lambda_{\min}(W_{\kappa i})\} \mathbb{E}\left[\int_0^t x^T(s)x(s) ds\right] \\ &\leq V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\ &\quad - \mathbb{E}\left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0)\right] \\ &\leq V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0).\end{aligned}$$

Thus,

$$\mathbb{E}\left[\int_0^t x^T(s)x(s) ds\right] \leq \frac{V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0)}{\min_{\kappa \in \mathcal{K}, i \in S} \{\lambda_{\min}(W_{\kappa i})\}}.$$

By letting  $t \rightarrow \infty$ ,

$$\mathbb{E}\left[\int_0^\infty x^T(s)x(s) ds\right] \leq \frac{V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0)}{\min_{\kappa \in \mathcal{K}, i \in S} \{\lambda_{\min}(W_{\kappa i})\}} < \infty.$$

Thus the system (5) is stochastically stable.  $\square$

*Remark 6.* Similar to remark 5, from step 0, given  $x(0) \in C_{i_0}$ , for  $i_0 \in \mathcal{K}$ , if  $\tau_0 = \infty$ , then the overall system (5) is equivalent to time-homogenous MJLS. And LMIs (8) are equivalent to the LMIs given in theorem 1 of [2] with the transition rate  $\lambda_{ij}^{i_0}$ .

### 3.2 Stochastic stabilization

In this section we provide a sufficient condition for stochastic stabilization with state feedback controller. Consider the system (5) with control input  $u(t) \in \mathbb{R}^m$

$$\dot{x}(t) = A_{r(\sigma_t, t)}x(t) + B_{r(\sigma_t, t)}u(t), \quad (10)$$

where  $B_{r(\sigma_t, t)} \in \mathbb{R}^{n \times m}$ .

We make use of the theorem 1 to design a state-feedback stabilizing controller such that the system (10) is stochastically stable. We assume that the system mode  $r(\sigma_t, t)$  is available in real time, which lead to a state feedback control law of the form

$$u(t) = K_{r(\sigma_t, t)}x(t), \quad K_{r(\sigma_t, t)} \in \mathbb{R}^{m \times n} \quad (11)$$

then the overall system is given by

$$\dot{x}(t) = \tilde{A}_{r(\sigma_t, t)}x(t), \quad (12)$$

where  $\tilde{A}_{r(\sigma_t, t)} = A_{r(\sigma_t, t)} + B_{r(\sigma_t, t)}K_{r(\sigma_t, t)}$ .

*Remark 7.* Since the system (12) is identical to the system (5), the existence and uniqueness of the solutions of (12) follow from remark 1.

The following theorem provides a sufficient condition for the existence of a stabilizing controller of the form (11).

**Theorem 2.** Consider the system (10) with  $\sigma_t$  and  $r(\sigma_t, t)$  described in (3) and (4). If there exist matrices  $X_i \succ 0$ , and  $Y_i$ , for each  $i \in S$  such that,

$$\begin{bmatrix} J_i & \mathcal{M}_{\kappa i} \\ * & -\mathcal{X}_i \end{bmatrix} \prec 0, \quad (13)$$

for each  $\kappa \in \mathcal{K}$ , where

$$\begin{aligned} J_i &= X_i A_i^T + Y_i^T B_i^T + A_i X_i + B_i Y_i + \left( \sum_{j=1}^K \lambda_{ii}^j I_{\{\kappa=j\}} \right) X_i, \\ \mathcal{M}_{\kappa i} &= [\sqrt{\lambda_{i1}^\kappa} X_i \cdots \sqrt{\lambda_{ii-1}^\kappa} X_i \sqrt{\lambda_{ii+1}^\kappa} X_i \cdots \sqrt{\lambda_{iN}^\kappa} X_i], \\ \mathcal{X}_i &= \text{diag}\{X_1 \cdots X_{i-1}, X_{i+1} \cdots X_N\}, \end{aligned}$$

then the system (10) is stochastically stabilized by (11), and the stabilizing controller is given by

$$K_i = Y_i X_i^{-1}. \quad (14)$$

*Proof.* The proof is an immediate extension of theorem 11 of [2].  $\square$

## 4 Illustrative Examples

In this section, two numerical examples are presented to illustrate the proposed results.

*Example 1.* Consider the SDJLS (1) with  $x(t) \triangleq [x_1(t), x_2(t)]^T \in \mathbb{R}^2$ . Let  $\theta(t) \in S := \{1, 2\}$ , be the state-dependent jump process given by (2), with  $C_1 \triangleq \{x(t) \in \mathbb{R}^2 : x_1^2(t) + x_2^2(t) < 3\}$ ,  $C_2 \triangleq \{x(t) \in \mathbb{R}^2 : x_1^2(t) + x_2^2(t) \geq 3\}$ , and for  $\mathcal{K} := \{1, 2\}$  the transition rate matrices of  $\theta(t)$  are given by

$$(\lambda_{ij}^1)_{2 \times 2} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}, (\lambda_{ij}^2)_{2 \times 2} = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}.$$

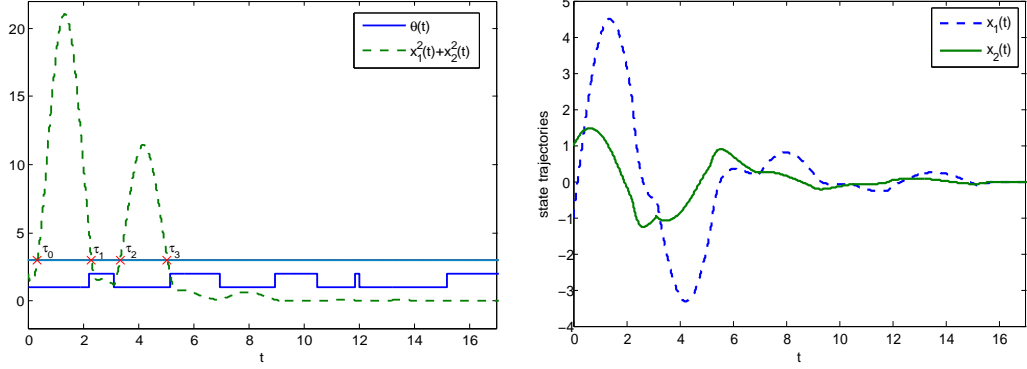
Let

$$A_1 = \begin{bmatrix} -1 & 5 \\ -0.5 & 0.9 \end{bmatrix}, A_2 = \begin{bmatrix} -4 & 2 \\ -2 & 0.1 \end{bmatrix}.$$

Then, from theorem 1, the LMIs (8), are satisfied with

$$P_1 = \begin{bmatrix} 0.3787 & -0.4069 \\ -0.4069 & 2.2977 \end{bmatrix}, P_2 = \begin{bmatrix} 0.3891 & -0.6203 \\ -0.6203 & 1.9226 \end{bmatrix}.$$

Hence, the system (1) is stochastically stable. With  $\theta(0) = 1$ ,  $x(0) = [-1, 1]^T$ , a sample  $\theta(t)$  with the corresponding stopping times  $\tau_0, \tau_1, \dots$  are given in step 0, step 1,  $\dots$  are plotted in figure 1a; the corresponding sample state trajectories of the system are shown in figure 1b.



(a) A sample  $\theta(t)$  and the corresponding stopping times (b) State trajectories of the system: a single sample path

Figure 1: Plots of example 1

*Example 2.* In this example we utilize the result of theorem 2 to synthesize a state feedback controller of form (11) that stochastically stabilizes (10). Let  $\theta(t) \in S := \{1, 2\}$  consists of the same parameters as example 1. Let

$$A_1 = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, B_2 = \begin{bmatrix} -5 \\ 6 \end{bmatrix}.$$

By solving the LMIs (13), we obtain:

$$X_1 = \begin{bmatrix} 0.343 & -0.365 \\ -0.365 & 0.3973 \end{bmatrix}, X_2 = \begin{bmatrix} 0.3998 & -0.4203 \\ -0.4203 & 0.4462 \end{bmatrix}, \\ Y_1 = \begin{bmatrix} 0.2597 & -0.5748 \end{bmatrix}, Y_2 = \begin{bmatrix} -0.0385 & 0.0032 \end{bmatrix},$$

and the feedback gains are obtained as:

$$K_1 = [-35.4961 \quad -34.0615], \quad K_2 = [-8.9022 \quad -8.3779].$$

With  $\theta(0) = 1$ ,  $x(0) = [-1, 1]^T$ , a single sample path simulation corresponding to a realization of  $\theta(t)$  is given in figure 2a; a sample state trajectories of the closed loop system resulting from the obtained controller are shown in figure 2b. It can be observed that the closed loop system is stochastically stable.

## 5 Conclusions

In this paper we have treated the stochastic stability and stabilization of a state-dependent jump linear system. We utilized the stopping times as a pointer to capture the evolution of the state variable, and used the Dynkin's formula to obtain sufficient condition in terms of linear matrix inequalities. Using the sufficient condition, we synthesize a state-feedback controller which stochastically stabilizes the system.

## Appendix

**Proof of Lemma 1:** We follow the approach used in [12] and [13] to prove. We prove in the sequel that  $(x(t), \theta(t))$  is jointly Markovian which is equivalent to state that  $(x(t), r(\sigma_t, t), \sigma_t)$  is a



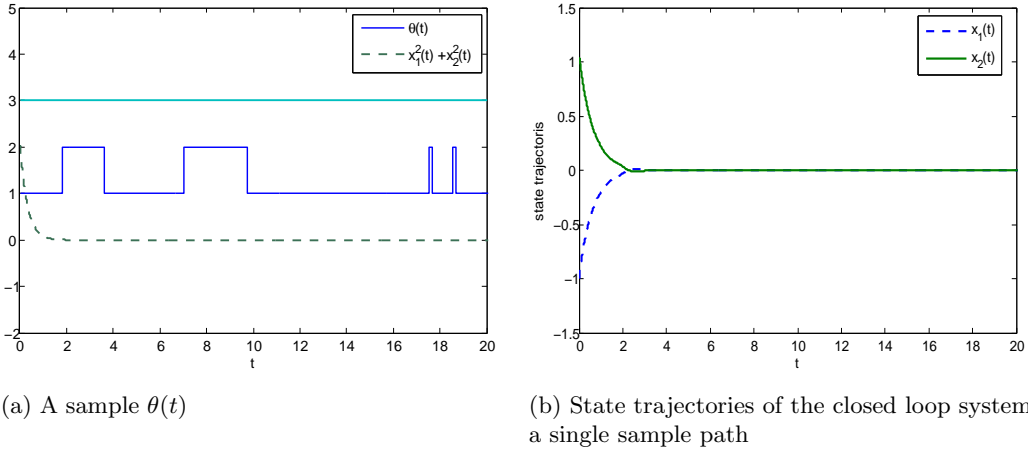


Figure 2: Plots of example 2

joint Markov process, because the evolution of  $\theta(t)$  is captured in  $r(\sigma_t, t)$  and  $\sigma_t$  from (3) and (4). Let  $\mathcal{F}_t$  denote the natural filtration of  $(x(t), \theta(t))$  on the interval  $[0, t]$ , and  $\mathcal{F}^t$  denote the natural filtration of  $(x(t), \theta(t))$  on the interval  $[t, \infty)$ . Let  $x(t)$  and  $\theta(t)$  from (1) and (2) be denoted, starting from  $t_0$ , as

$$x(t) \triangleq x_t^{t_0, x(t_0), \theta(t_0)}, \quad \theta(t) \triangleq \theta_t^{t_0, x(t_0), \theta(t_0)}.$$

Denote the joint process  $\xi_t^{t_0, \eta(t_0)}$  starting from  $t_0$  as

$$\xi_t^{t_0, \eta(t_0)} \triangleq \left( x_t^{t_0, x(t_0), \theta(t_0)}, \theta_t^{t_0, x(t_0), \theta(t_0)} \right),$$

with  $\eta(t_0) = \left( x_{t_0}^{t_0, x(t_0), \theta(t_0)}, \theta_{t_0}^{t_0, x(t_0), \theta(t_0)} \right)$ , which is simply  $(x(t_0), \theta(t_0))$ . For  $0 < s < t$ ,  $\xi_t^{s, \eta(s)}$  describes the process on  $[s, \infty)$  with  $\eta(s) = \left( x_s^{s, x(s), \theta(s)}, \theta_s^{s, x(s), \theta(s)} \right)$ , thus it is  $\mathcal{F}^s$ -measurable. Let  $\eta(s)$  be an arbitrary  $\mathcal{F}_s$ -measurable random variable. For  $0 < \tau < s < t$ ,  $\xi_t^{\tau, \eta(\tau)}$  can be described as a  $\mathcal{F}^s$ -measurable process on  $[s, \infty)$  with initial condition  $\xi_s^{\tau, \eta(\tau)}$ . Thus we can write

$$\xi_t^{\tau, \eta(\tau)} = \xi_t^{s, \xi_s^{\tau, \eta(\tau)}}, \quad \text{for } 0 < \tau < s < t. \quad (15)$$

Let  $B$  be a set in the  $\sigma$ -algebra of Borel sets on  $\mathbb{R}^n \times S$ . Then

$$\begin{aligned} Pr\{\xi_t^{\tau, \eta(\tau)} \in B | \mathcal{F}_s\} &= E \left[ I_B \left( \xi_t^{\tau, \eta(\tau)} \right) | \mathcal{F}_s \right] = E \left[ I_B \left( \xi_t^{s, \xi_s^{\tau, \eta(\tau)}} \right) | \mathcal{F}_s \right] \\ &= E \left[ I_B \left( \xi_t^{s, \eta(s)} \right) \right]_{\eta(s) = \xi_s^{\tau, \eta(\tau)}} \\ &= Pr\{\xi_t^{s, \eta(s)} \in B\}, \end{aligned}$$

which completes the proof.  $\square$

**Proof of Lemma 2:** Let  $\mathcal{G}_t$  denote the natural filtration of  $(x(t), r(\sigma_t, t))$  on the interval  $[0, t]$ . Consider

$$\begin{aligned} \{\tau_0 \leq t\} &= \bigcup_{s \in \mathbb{Q} \cap [0, t]} \{\Phi_{i_0}(s, 0)x(0) \notin C_{i_0}\} \\ &= \Omega \setminus \bigcap_{s \in \mathbb{Q} \cap [0, t]} \{\Phi_{i_0}(s, 0)x(0) \in C_{i_0}\}. \end{aligned}$$

From the above argument, observe that, each event  $\{\Phi_{i_0}(s, 0)x(0) \in C_{i_0}\}$  is  $\mathcal{G}_t$ -measurable for all  $s \in \mathbb{Q} \cap [0, t]$ . Consequently the event  $\{\tau_0 \leq t\}$  is also  $\mathcal{G}_t$ -measurable, as the complement of the intersection of  $\mathcal{G}_t$ -measurable events are also  $\mathcal{G}_t$ -measurable. Thus  $\tau_0$  is a stopping time. The similar arguments are applied to  $\tau_1, \tau_2, \dots$ .  $\square$

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